On the Ternary Quadratic Diophantine Equation $z^2 = 7x^2 + y^2$

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Abstract – This paper concerns with the problem of obtaining a general solution of the equation $z^2 = 7x^2 + y^2$ based on its given initial solution.

Index Terms – Ternary quadratic, Homogeneous cone, integer solutions.

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1. INTRODUCTION

The quadratic Diophantine equations with three unknowns offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems on ternary quadratic Diophantine equations representing specific 3 dimensional surfaces, one may refer [3-24].

In this communication, we present a problem of obtaining a general solution of the equation $z^2 = 7x^2 + y^2$ based on its given initial solution.

2. METHOD OF ANALYSIS

The ternary quadratic Diophantine equation under consideration is

$$z^2 = 7x^2 + y^2 \tag{1}$$

To start with, applying the method of factorization and performing a few calculations, it is seen that equation (1) is satisfied by the triples of integers$(k, 3k, 4k), (2k + 1, 2k^2 + 2k - 3, 2k^2 + 2k + 4)$ and $(2k + 1, 14k^2 + 14k + 3, 14k^2 + 14k + 4)$

A natural question that arises now is that, whether a general formula for obtaining a sequence of integer solutions for (1) based on its given integer solution can be found?

The answer to this question is yes. And a method of obtaining the same is illustrated below.

Let (x_0, y_0, z_0) be the given integer solution to (1). Let (x_1, y_1, z_1) be the first solution of (1) where

$$x_1 = x_0 + h, y_1 = y_0 + h, z_1 = 3h - z_0 \tag{2}$$

Where h is any non- zero integer to be determined

Substituting (2) in (1) and simplifying, we have

$$h = 14x_0 + 2y_0 + 6z_0 \tag{3}$$

Using (3) in (2), we have the values of x, y, z represented in the matrix form as

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} M \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\text{Where } \begin{pmatrix} M \end{pmatrix} = \begin{pmatrix} 15 & 2 & 6 \\ 14 & 3 & 6 \\ 42 & 6 & 17 \end{pmatrix}$$

Repeating the above process, the general solution $(x_{k+1}, y_{k+1}, z_{k+1})$ of (1) is represented in the matrix form as below

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} y_{2k+1} - 2x_k & \frac{y_{2k+1} - 2x_k - 1}{7} & x_{2k+1} \\ y_{2k+1} - 2x_k - 1 & \frac{y_{2k+1} - 2x_k + 6}{7} & x_{2k+1} \\ 7x_{2k+1} & x_{2k+1} & y_{2k+1} \end{pmatrix}; k \geq 0$$

Where (x_{2k+1}, y_{2k+1}) is the solution of the Pell equation $y^2 = 8x^2 + 1$ with initial solution$x_0 = 1, y_0 = 3$.

3. REMARK

It is worth to mention here that the general solution obtained above for equation (1) is not unique. In particular, we have two more choices of general solutions to equation (1) that are presented below. Let (x_0, y_0, z_0) be any given solution of (1)

3.1 CHOICE 1

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 8^{n-1} \begin{pmatrix} 7 + (-1)^n & 1 - (-1)^n \\ 7 - (-1)^n & 1 + 7(-1)^n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}; z_n = 8^n z_0$$

3.2 CHOICE 2

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} Z_{n-1} & X_{n-1} \\ 7X_{n-1} & Z_{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}; N = 1,2,3 \ldots$$

where (X_n, Z_n) is the general solution of the Pellian $Z^2 = 7X^2 + 1$ and $y_n = y_0$

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